

Algebraic Bethe Ansatz for XYZ Gaudin model

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Dedicated to Professor Hikosaburo Komatsu on his 60th birthday

Abstract

The eigenvectors of the Hamiltonians of the XYZ Gaudin model are constructed by means of the algebraic Bethe Ansatz. The construction is based on the quasi-classical limit of the corresponding results for the inhomogeneous higher spin eight vertex model.

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1 Introduction

The XYZ Gaudin model was introduced by M. Gaudin [1, 2, 3] as a quasiclassical limit of XYZ spin-1/2 chain. Gaudin noticed also that the former model can be generalized to any values of constituting spins. Whereas the spectrum and eigenfunctions of the XXX and XXZ variants of Gaudin model can easily be found via Bethe ansatz, the general case encounters the same problems as in case of the original XYZ model. For the spin-1/2 XYZ model a proper generalization of Bethe ansatz was found in [4], see also [5]. However, when passing to the Gaudin model, an additional problem arises: whereas it is easy to perform quasiclassical limit in the Bethe equations determining the spectrum of XYZ model, the expressions for the eigenvectors appear to behave singularly. Solution to this problem is the main subject of the present paper.

We obtain the arbitrary spin XYZ Gaudin model as a quasiclassical limit of the inhomogeneous higher spin generalization of XYZ model introduced in [6, 7, 8]. To construct the eigenvectors of XYZ model we use the algebraic version of Bethe ansatz proposed in [5] for spin-1/2 and generalized to higher spins in [6, 7, 8]. The paper is concluded with describing a way to circumvent the divergencies in the quasiclassical limit and to get finite expressions for Bethe eigenvectors of Gaudin model.

2 Inhomogeneous XYZ model

First let us briefly recall the inhomogeneous XYZ spin chain model. The model is parametrized by an anisotropy parameter η besides the elliptic modulus τ , shifts of the spectral parameter z_n , and a spin at each site ℓ_n . Later we will take the limit $\eta \rightarrow 0$ and recover the generating function of the integrals of motion of the XYZ Gaudin model as the leading term of the transfer matrix of this model as in [9].

The inhomogeneous XYZ model is a quantum spin chain model defined on the Hilbert space $\mathcal{H} = \bigotimes_{n=1}^N V_n$, where V_n is the spin ℓ_n representation space $\Theta_{00}^{4\ell_n+}$ of the Sklyanin algebra [10], [11]. See Appendix A for notations.

The transfer matrix of the model, $\hat{t}(u)$, is a linear operator on \mathcal{H} defined as the trace of the monodromy matrix, $T(u)$, in the context of the quantum inverse scattering method [5]:

$$T(u) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix} := L_N(u - z_N) \dots L_2(u - z_2) L_1(u - z_1) \quad (2.1)$$

$$\hat{t}(u) = \text{tr}_{\mathbf{C}^2}(T(u)) = A_N(u) + D_N(u), \quad (2.2)$$

where the L operators, $L_n(u)$, whose elements act non-trivially only on the n -th

component, V_n , are defined by (cf. (A.2))

$$L_n(u) = \begin{pmatrix} \alpha_n(u) & \beta_n(u) \\ \gamma_n(u) & \delta_n(u) \end{pmatrix} := \sum_{a=0}^3 W_a^L(u) \rho_n^{\ell_n}(S^a) \otimes \sigma^a, \quad (2.3)$$

$$\rho_n^{\ell_n}(S^a) = 1 \otimes \dots \otimes 1 \otimes \rho_n^{\ell_n}(S^a) \otimes 1 \otimes \dots \otimes 1. \quad (2.4)$$

Because of the fundamental commutation relation,

$$R_{12}(u_1 - u_2) T_{01}(u_1) T_{02}(u_2) = T_{02}(u_2) T_{01}(u_1) R_{12}(u_1 - u_2), \quad (2.5)$$

the transfer matrices commute with each other:

$$[\hat{t}(u_1), \hat{t}(u_2)] = 0. \quad (2.6)$$

The quantum determinant ([12]) is defined by

$$\Delta(u) := \text{tr}_{12} P_{12}^- T_{01}(u - \eta) T_{02}(u + \eta), \quad (2.7)$$

where P_{12}^- is a projector onto the space of anti-symmetric tensors. This operator commute with all elements of the Sklyanin algebra and acts on the Hilbert space \mathcal{H} as a scalar multiplication:

$$\Delta(u)|_{\mathcal{H}} = \Delta_-(u - \eta) \Delta_+(u + \eta), \quad (2.8)$$

$$\Delta_{\pm}(u) = \prod_{n=1}^N \frac{\theta_{11}(u - z_n \pm 2\ell_n \eta)}{\theta_{11}(u - z_n)}. \quad (2.9)$$

3 XYZ Gaudin model

We define the XYZ Gaudin model as a quasi-classical limit of the inhomogeneous XYZ model defined above. Let us examine the asymptotic behaviour of the operators in the previous section when η tends to 0. The L operator, the monodromy matrix, the transfer matrix, the quantum determinant and the R matrix are expanded as

$$L_n(u) = 1 + 2\eta \mathcal{L}_n(u) + O(\eta^2), \quad (3.1)$$

$$T(u) = 1 + 2\eta \mathcal{T}(u) + \eta^2 \mathcal{T}^{(2)}(u) + O(\eta^2), \quad (3.2)$$

$$\hat{t}(u) = 1 + \eta^2 \text{tr} \mathcal{T}^{(2)}(u) + O(\eta^3), \quad (3.3)$$

$$\Delta(u) = 1 + \eta^2 (\text{tr} \mathcal{T}^{(2)}(u) + 4 \text{tr} \mathcal{T}(u)^2) + O(\eta^3), \quad (3.4)$$

$$R(u) = 1 - 2\eta r(u) + O(\eta^2). \quad (3.5)$$

Explicitly

$$\mathcal{L}_n(u) = \sum_{a=1}^3 w_a(u) \rho_n^{\ell_n} S^a \otimes \sigma^a, \quad \mathcal{T}(u) = \sum_{n=1}^N \mathcal{L}_n(u - z_n).$$

Here ρ^ℓ denotes the spin ℓ representations of the Lie algebra $sl(2)$, and

$$\begin{aligned} w_1(u) &= \frac{\sigma_{10}(u)}{\sigma(u)} = \frac{\text{cn}}{\text{sn}}(u) = \frac{\theta'_{11}}{\theta_{10}} \frac{\theta_{10}(u)}{\theta_{11}(u)}, \\ w_2(u) &= \frac{\sigma_{00}(u)}{\sigma(u)} = \frac{\text{dn}}{\text{sn}}(u) = \frac{\theta'_{11}}{\theta_{00}} \frac{\theta_{00}(u)}{\theta_{11}(u)}, \\ w_3(u) &= \frac{\sigma_{01}(u)}{\sigma(u)} = \frac{1}{\text{sn}(u)} = \frac{\theta'_{11}}{\theta_{01}} \frac{\theta_{01}(u)}{\theta_{11}(u)}, \end{aligned}$$

where $\theta_{ab} = \theta_{ab}(0)$, $\theta'_{11} = d/du(\theta_{11}(u))|_{u=0}$, σ is Weierstraß' sigma function and \mathcal{S}^a are generators of the Lie algebra $sl(2)$:

$$[\mathcal{S}^a, \mathcal{S}^b] = i\mathcal{S}^c.$$

Here (a, b, c) denotes a cyclic permutation of $(1, 2, 3)$. The commutation relations of the \mathcal{L} and the monodromy operator are

$$[\mathcal{L}_{01}(u), \mathcal{L}_{02}(v)] = [r(u-v), \mathcal{L}_{01}(u) + \mathcal{L}_{02}(v)], \quad (3.6)$$

$$[\mathcal{T}_{01}(u), \mathcal{T}_{02}(v)] = [r(u-v), \mathcal{T}_{01}(u) + \mathcal{T}_{02}(v)]. \quad (3.7)$$

The *classical r matrix* r is defined by (3.5), or explicitly by

$$r(u) = -\frac{1}{2} \sum_{a=1}^3 w_a(u) \sigma^a \otimes \sigma^a.$$

The commutation relation (3.6) is nothing but the quasi-classical limit of the fundamental commutation relation (2.5).

We define the *XYZ Gaudin model* by specifying its generating function of integrals of motion as $\hat{\tau}(u) = \text{tr } \mathcal{T}(u)^2$. It is obvious from (3.3) and (3.4) that $\hat{\tau}(u)$ can be expressed as follows (cf. [9]):

$$\Delta(u) - \hat{t}(u) - 1 = 4\eta^2 \hat{\tau}(u) + O(\eta^3). \quad (3.8)$$

Therefore we can expect that eigenvectors of $\hat{\tau}(u)$ can be constructed as a leading term of the η expansion of eigenvectors of $\hat{t}(u)$ which are found in [6], [7], [8]. Essentially this is true, but we must be careful in taking the limit as we will see in the next section.

Operator $\hat{\tau}(u)$ is explicitly written down as follows:

$$\hat{\tau}(u) = \sum_{n=1}^N \wp(u - z_n) \ell_n(\ell_n + 1) + \sum_{n=1}^N H_n \zeta(u - z_n) + H_0, \quad (3.9)$$

where ζ and \wp are Weierstraß' zeta and \wp functions and

$$H_n = 2 \sum_{m \neq n} \sum_{a=1}^3 w_a(z_n - z_m) \mathcal{S}_n^a \mathcal{S}_m^a, \quad \sum_{n=1}^N H_n = 0, \quad (3.10)$$

$$\begin{aligned}
H_0 &= - \sum_{n=1}^N \sum_{a=1}^3 \left(e_a (\mathcal{S}_n^a)^2 + \sum_{m \neq n} w_a (z_n - z_m) \int_{\omega_{\bar{a}}/2}^{z_n - z_m + \omega_{\bar{a}}/2} \wp(u) du \mathcal{S}_n^a \mathcal{S}_m^a \right) \\
&= \sum_{n=1}^N \sum_{a=1}^3 \left(-e_a (\mathcal{S}_n^a)^2 + \right. \\
&\quad \left. + \sum_{m \neq n} w_a (z_n - z_m) \left(\zeta \left(z_n - z_m + \frac{\omega_{\bar{a}}}{2} \right) - \zeta \left(\frac{\omega_{\bar{a}}}{2} \right) \right) \mathcal{S}_n^a \mathcal{S}_m^a \right) \quad (3.11)
\end{aligned}$$

are integrals of motion. Here \bar{a} is 1, 3, 2 for $a = 1, 2, 3$ respectively, $\omega_1 = 1$, $\omega_2 = \tau$, $\omega_3 = 1 + \tau$, $e_a = \wp(\omega_{\bar{a}}/2)$. We omit the symbol ρ^ℓ for the sake of simplicity.

4 Algebraic Bethe Ansatz for XYZ spin chain

The algebraic Bethe Ansatz in the context of the quantum inverse scattering method is applied to the inhomogeneous XYZ spin chain model in the following way. (cf. [5], [6], [8]), Hereafter we assume that the total spin $\ell_{\text{total}} = \ell_1 + \dots + \ell_N$ is equal to an integer M .

First we introduce the matrix of the gauge transformation:

$$M_\lambda(u) := \begin{pmatrix} -\theta_{01} \left(\frac{\lambda-u}{2}; \frac{\tau}{2} \right) & -\theta_{01} \left(\frac{\lambda+u}{2}; \frac{\tau}{2} \right) \\ \theta_{00} \left(\frac{\lambda-u}{2}; \frac{\tau}{2} \right) & \theta_{00} \left(\frac{\lambda+u}{2}; \frac{\tau}{2} \right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \theta_{11}(\lambda; \tau)^{-1} \end{pmatrix}. \quad (4.1)$$

A twisted monodromy matrix is defined by means of M_λ :

$$\begin{aligned}
T_{\lambda, \lambda'}(u; v) &= \begin{pmatrix} A_{\lambda, \lambda'}(u; v) & B_{\lambda, \lambda'}(u; v) \\ C_{\lambda, \lambda'}(u; v) & D_{\lambda, \lambda'}(u; v) \end{pmatrix} \\
&:= M_\lambda(u+v)^{-1} T(u) M_{\lambda'}(u+v), \quad (4.2)
\end{aligned}$$

where v is a parameter. There is a vector $\omega_\lambda(v)$ called a local pseudovacuum in the spin ℓ representation space $V_\ell = \Theta_{00}^{4\ell+}$:

$$\begin{aligned}
\omega_\lambda(v) = \omega_\lambda(v; y) &:= \prod_{j=1}^{2\ell} \theta_{10} \left(y + \frac{\lambda}{2} - \frac{v}{2} + (2j - 3\ell - 1)\eta \right) \times \\
&\times \theta_{10} \left(y - \frac{\lambda}{2} + \frac{v}{2} - (2j - 3\ell - 1)\eta \right),
\end{aligned}$$

and a global pseudovacuum in \mathcal{H} :

$$\begin{aligned}
\Omega_N^\lambda(v) &:= \omega_{\lambda+4(\ell_1+\dots+\ell_N)\eta}(v+z_N) \otimes \dots \\
&\dots \otimes \omega_{\lambda+4(\ell_1+\dots+\ell_n)\eta}(v+z_n) \otimes \dots \otimes \omega_{\lambda+4\ell_1\eta}(v+z_1). \quad (4.3)
\end{aligned}$$

As shown in [5] and [6], we can construct eigenvectors of the transfer matrix $\hat{t}(u)$ of the form

$$\Psi_\nu(w_1, \dots, w_M) = \sum_{a \in \mathbf{Z}} e^{2\pi i \nu a \eta} \Phi_{\lambda+2a\eta}(w_1, \dots, w_M), \quad (4.4)$$

$$\begin{aligned} \Phi_\lambda(w_1, \dots, w_M; v) &:= B_{\lambda+2\eta, \lambda-2\eta}(w_1) B_{\lambda+4\eta, \lambda+4\eta}(w_2) \dots \\ &\dots B_{\lambda+2M\eta, \lambda-2M\eta}(w_M) \Omega_N^{\lambda-2M\eta}(v), \end{aligned} \quad (4.5)$$

where an integer ν and the parameters w_1, \dots, w_M satisfy a system of equations (Bethe equations) for $j = 1, \dots, M$:

$$\frac{\Delta_+(w_j)}{\Delta_-(w_j)} = e^{-4\pi i \nu \eta} \prod_{k=1, k \neq j}^M \frac{\theta_{11}(w_j - w_k + 2\eta)}{\theta_{11}(w_j - w_k - 2\eta)}. \quad (4.6)$$

The eigenvalue $t(u)$ of the transfer matrix $\hat{t}(u)$:

$$\hat{t}(u) \Psi_\nu(w_1, \dots, w_M) = t(u) \Psi_\nu(w_1, \dots, w_M), \quad (4.7)$$

satisfies

$$t(u)q(u) = \Delta_+(u)q(u-2\eta) + \Delta_-(u)q(u+2\eta), \quad (4.8)$$

where

$$q(u) = e^{-\pi i \nu u} \prod_{m=1}^M \theta_{11}(u - w_m). \quad (4.9)$$

Note that the Bethe vector $\Psi_\nu(w_1, \dots, w_M)$ is expressed as a Fourier series, and the convergence of such series is not a priori known. In fact when $\eta = r'/r$ is a rational number, this Fourier series diverges and we must replace it by a sum with r terms. Hence we cannot take the limit $\eta \rightarrow 0$ of the above results naively to obtain eigenvectors of the XYZ Gaudin model.

5 Algebraic Bethe Ansatz for XYZ Gaudin model

In order to obtain an eigenvector of the XYZ Gaudin model, we bypass the divergent series (4.4) in the following way. At an intermediate stage of derivation of the Bethe equations, we have the formula:

$$\begin{aligned} \hat{t}(u) \Phi_\lambda(\vec{w}; v) &= \Lambda(u; \vec{w}; \eta) \Phi_{\lambda-2\eta}(\vec{w}; v) + \Lambda(u; \vec{w}; -\eta) \Phi_{\lambda+2\eta}(\vec{w}; v) + \\ &+ \sum_{j=1}^M \Lambda_j^\lambda(u; \vec{w}; \eta) \Phi_{\lambda-2\eta}(\vec{w}_j(u); v) + \sum_{j=1}^M \Lambda_j^\lambda(u; \vec{w}; -\eta) \Phi_{\lambda+2\eta}(\vec{w}_j(u); v) \end{aligned} \quad (5.1)$$

where we abbreviated (w_1, \dots, w_M) to \vec{w} and $(w_1, \dots, w_{j-1}, u, w_{j+1}, \dots, w_M)$ to $\vec{w}_j(u)$, and Λ and Λ_j^λ are defined by:

$$\Lambda(u; \vec{w}; \eta) = \Delta_+(u) \prod_{m=1}^M \frac{\theta_{11}(u - w_m - 2\eta)}{\theta_{11}(u - w_m)} \quad (5.2)$$

$$\Lambda_j^\lambda(u; \vec{w}; \eta) = \theta_{11}(-2\eta) \frac{\theta_{11}((u - w_j) - (\lambda - 2\eta))}{\theta_{11}(u - w_j) \theta_{11}(\lambda - 2\eta)} \times \quad (5.3)$$

$$\times \Delta_+(w_j) \prod_{k=1, k \neq j}^M \frac{\theta_{11}(w_j - w_k - 2\eta)}{\theta_{11}(w_j - w_k)}. \quad (5.4)$$

Since all of these formulas are holomorphic around $\eta = 0$, we can expand them as Taylor series in η . For example, as $M_\lambda(u)$ does not depend on η and the monodromy matrix $T(u)$ has the expansion (3.2), the twisted monodromy matrix $T_{\lambda+2m\eta, \lambda-2m\eta}(u; v)$ has the expansion,

$$T_{\lambda+2m\eta, \lambda-2m\eta}(u; v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(\eta). \quad (5.5)$$

Therefore (1,2)-element of the twisted monodromy matrix has an expansion as follows:

$$B_{\lambda+2m\eta, \lambda-2m\eta}(u; v) = \eta \mathcal{B}_{\lambda, m}(u, v) + O(\eta^2). \quad (5.6)$$

The pseudovacuum vector defined by (4.3) is expanded as

$$\Omega_N^{\lambda-2M\eta}(v) = \Omega_N^{\lambda, M, (0)}(v) + \eta \Omega_N^{\lambda, M, (1)}(v) + \dots \quad (5.7)$$

Combining (5.6) and (5.7), we obtain

$$\Phi_\lambda(\vec{w}; v) = \eta^M (\phi_\lambda(\vec{w}; v) + O(\eta)). \quad (5.8)$$

It is possible to write down the complicated definition of $\phi_\lambda(\vec{w}; v)$ explicitly, but it is not necessary to our purpose here. Expanding Λ (5.2) and Λ_j^λ (5.4) as

$$\Lambda(u; \vec{w}; \eta) = 1 + \eta \Lambda^{(1)}(u; \vec{w}) + \eta^2 \Lambda^{(2)}(u; \vec{w}) + O(\eta^3), \quad (5.9)$$

$$\Lambda_j^\lambda(u; \vec{w}; \eta) = \eta \Lambda_j^{\lambda(1)}(u; \vec{w}) + \eta^2 \Lambda_j^{\lambda(2)}(u; \vec{w}) + O(\eta^3), \quad (5.10)$$

we obtain the following equation for the action of leading terms of $\hat{t}(u)$ on $\phi_\lambda(u; \vec{w}; v)$ from (5.1) and (3.3):

$$\begin{aligned} & \text{tr } \mathcal{T}^{(2)}(u) \phi_\lambda(\vec{w}; v) \\ &= \left(4 \frac{\partial^2}{\partial \lambda^2} - 4 \Lambda^{(1)}(u; \vec{w}) \frac{\partial}{\partial \lambda} + 2 \Lambda^{(2)}(u; \vec{w}) \right) \phi_\lambda(\vec{w}; v) \\ &+ \sum_{j=1}^M \left(-4 \Lambda_j^{\lambda(1)}(u; \vec{w}) \frac{\partial}{\partial \lambda} + 2 \Lambda_j^{\lambda(2)}(u; \vec{w}) \right) \phi_\lambda(\vec{w}_j(u); v). \end{aligned} \quad (5.11)$$

Instead of the Fourier series (4.4), we take a Fourier transformation of $\phi_\lambda(\vec{w}; v)$ as a candidate of an eigenvector of $\text{tr } \mathcal{T}^{(2)}(u)$ (and hence of $\hat{\tau}(u)$):

$$\psi_\nu(\vec{w}) := \int_{-1}^1 e^{\pi i \nu \lambda} \phi_\lambda(\vec{w}; v) d\lambda. \quad (5.12)$$

Here ν is an integer to be determined. Since $\phi_\lambda(\vec{w}; v)$ is periodic with respect to λ with period 2, operator $\text{tr } \mathcal{T}^{(2)}$ acts on ψ_ν as follows by virtue of (5.11):

$$\begin{aligned} \text{tr } \mathcal{T}^{(2)}(u) \psi_\nu(\vec{w}) &= (4(\pi i \nu)^2 + 4\pi i \nu \Lambda^{(1)}(u; \vec{w}) + 2\Lambda^{(2)}(u; \vec{w})) \psi_\nu(\vec{w}) \\ &+ \sum_{j=1}^M (\text{unwanted})_j, \end{aligned} \quad (5.13)$$

Here “unwanted terms” are defined by

$$\begin{aligned} (\text{unwanted})_j &= \int_{-1}^1 e^{\pi i \nu \lambda} \left(-4\Lambda_j^{\lambda(1)}(u; \vec{w}) \frac{\partial}{\partial \lambda} + 2\Lambda_j^{\lambda(2)}(u; \vec{w}) \right) \phi_\lambda(\vec{w}_j(u); v) d\lambda \\ &= \int_{-1}^1 e^{\pi i \nu \lambda} \left(\pi i \nu + \sum_{n=1}^N \ell_n \frac{\theta'_{11}(w_j - z_n)}{\theta_{11}(w_j - z_n)} - \sum_{k=1, k \neq j}^M \frac{\theta'_{11}(w_j - w_k)}{\theta_{11}(w_j - w_k)} \right) \times \\ &\times 4\Lambda_j^{\lambda(1)}(u; \vec{w}) \phi_\lambda(\vec{w}_j(u); v) d\lambda, \end{aligned} \quad (5.14)$$

where $'$ denotes the derivative. Thus, if (...) in the last expression of (5.14) vanishes, namely, if

$$\sum_{n=1}^N \ell_n \frac{\theta'_{11}(w_j - z_n)}{\theta_{11}(w_j - z_n)} = -\pi i \nu + \sum_{k=1, k \neq j}^M \frac{\theta'_{11}(w_j - w_k)}{\theta_{11}(w_j - w_k)} \quad (5.15)$$

holds for any $j = 1, \dots, M$, then (5.13) implies that $\psi_\nu(w_1, \dots, w_M)$ is an eigenvector of $\text{tr } \mathcal{T}^{(2)}$.

The corresponding eigenvalue of $\hat{\tau}(u)$ can be computed from the expansion (5.9), but a simpler way is the following: Observe that the eigenvalue of $\text{tr } \mathcal{T}^{(2)}(u)$ for $\psi_\nu(\vec{w})$

$$4(\pi i \nu)^2 + 4\pi i \nu \Lambda^{(1)}(u; \vec{w}) + 2\Lambda^{(2)}(u; \vec{w})$$

is equal to the coefficient at η^2 of the right hand side of (4.8) in which we replace $\{w_j\}$ by a solution to (5.15) this time. Thus we can derive the formula for an eigenvalue of $\hat{\tau}(u)$ corresponding to ψ_ν by extracting terms of order η^2 in (2.8) and (3.4) (cf. [9]):

$$q''(u) - 2Z(u)q'(u) + (Z(u)^2 - Z(u)')q(u) = \tau(u)q(u), \quad (5.16)$$

or

$$\tau(u) = (\chi(u) - Z(u))^2 + \frac{d}{du}(\chi(u) - Z(u)), \quad (5.17)$$

where

$$Z(u) := \sum_{n=1}^N \ell_n \frac{\theta'_{11}(u - z_n)}{\theta_{11}(u - z_n)} \quad \chi(u) := \frac{q'(u)}{q(u)} = \sum_{m=1}^M \frac{\theta'_{11}(u - w_m)}{\theta_{11}(u - w_m)}.$$

This result is an elliptic analogue of the spectrum of the XXX Gaudin model [1], [2], [3], [9].

6 Comments

We applied the algebraic Bethe Ansatz to the XYZ Gaudin model and constructed eigenvectors of the generating function of integrals of motion, $\hat{\tau}(u)$. Several questions need, however, further investigation.

First, it would be preferable to have a closed treatment of the Gaudin model, completely independent of the original XYZ model. The difficulty is that the expression for $\phi_\lambda(\vec{w}; v)$ obtained by differentiation of $\Phi_\lambda(\vec{w}; v)$ is rather voluminous.

Another weak point is that we must assume that the total spin ℓ_{total} must be an integer. This obstruction comes from the same assumption in the Bethe Ansatz of the XYZ type spin chains [5], [6].

It is also an open question if we can obtain all the eigenvectors within our approach. However, this is quite common situation in the application of the algebraic Bethe Ansatz known as completeness problem.

We expect that these problems would be overcome within an alternative approach to the model known as Separation of Variables, [13]. The functional equation (5.16) should be interpreted then as a separated equation.

A Review of the Sklyanin algebra

In this appendix we recall several facts on the Sklyanin algebra and its representations from [10] and [11]. We use the notation from [14] for theta functions:

$$\theta_{ab}(z; \tau) = \sum_{n \in \mathbf{Z}} \exp \left(\pi i \left(\frac{a}{2} + n \right)^2 \tau + 2\pi i \left(\frac{a}{2} + n \right) \left(\frac{b}{2} + z \right) \right),$$

where τ is a complex number in the upper half plane.

The *Sklyanin algebra*, $U_{\tau, \eta}(sl(2))$ is generated by four generators S^0, S^1, S^2, S^3 , satisfying the following relations:

$$R_{12}(u - v) L_{01}(u) L_{02}(v) = L_{02}(v) L_{01}(u) R_{12}(u - v). \quad (\text{A.1})$$

Here u, v are complex parameters, the *L operator*, $L(u)$, is defined by

$$L(u) = \sum_{a=0}^3 W_a^L(u) S^a \otimes \sigma^a, \quad (\text{A.2})$$

$$\begin{aligned}
W_0^L(u) &= \frac{1}{2\theta_{11}(\eta; \tau)}, & W_1^L(u) &= \frac{\theta_{10}(u; \tau)}{2\theta_{11}(u; \tau)\theta_{10}(\eta; \tau)}, \\
W_2^L(u) &= \frac{\theta_{00}(u; \tau)}{2\theta_{11}(u; \tau)\theta_{00}(\eta; \tau)}, & W_3^L(u) &= \frac{\theta_{01}(u; \tau)}{2\theta_{11}(u; \tau)\theta_{01}(\eta; \tau)},
\end{aligned}$$

$R(u)$ is *Baxter's R matrix* defined by

$$R(u) = \sum_{a=0}^3 W_a^R(u) \sigma^a \otimes \sigma^a, \quad W_a^R(u) := W_a^L(u + \eta) / W_0^L(u + \eta). \quad (\text{A.3})$$

and indices $\{0, 1, 2\}$ denote the spaces on which operators act non-trivially: for example,

$$R_{12}(u) = \sum_{a=0}^3 W_a^R(u) 1 \otimes \sigma^a \otimes \sigma^a, \quad L_{01}(u) = \sum_{a=0}^3 W_a^L(u) S^a \otimes \sigma^a \otimes 1.$$

The above relation (A.1) contains u and v as parameters, but the commutation relations among S^a ($a = 0, \dots, 3$) do not depend on them:

$$[S^\alpha, S^0]_- = -iJ_{\alpha, \beta}[S^\beta, S^\gamma]_+, \quad [S^\alpha, S^\beta]_- = i[S^0, S^\gamma]_+, \quad (\text{A.4})$$

where (α, β, γ) stands for any cyclic permutation of $(1, 2, 3)$, $[A, B]_\pm = AB \pm BA$, and $J_{\alpha, \beta} = (W_\alpha^2 - W_\beta^2) / (W_\gamma^2 - W_0^2)$, i.e.,

$$J_{12} = \frac{\theta_{01}(\eta; \tau)^2 \theta_{11}(\eta; \tau)^2}{\theta_{00}(\eta; \tau)^2 \theta_{10}(\eta; \tau)^2}, \quad J_{23} = \frac{\theta_{10}(\eta; \tau)^2 \theta_{11}(\eta; \tau)^2}{\theta_{00}(\eta; \tau)^2 \theta_{01}(\eta; \tau)^2}, \quad J_{31} = -\frac{\theta_{00}(\eta; \tau)^2 \theta_{11}(\eta; \tau)^2}{\theta_{01}(\eta; \tau)^2 \theta_{10}(\eta; \tau)^2}.$$

The *spin ℓ representation* of the Sklyanin algebra,

$$\rho^\ell : U_{\tau, \eta}(sl(2)) \rightarrow \text{End}_{\mathbb{C}}(\Theta_{00}^{4\ell})$$

is defined as follows: The representation space is

$$\Theta_{00}^{4\ell+} = \{f(z) \mid f(z+1) = f(-z) = f(z), f(z+\tau) = \exp^{-4\ell\pi i(2z+\tau)} f(z)\}.$$

It is easy to see that $\dim \Theta_{00}^{4\ell+} = 2\ell + 1$. The generators of the algebra act on this space as difference operators:

$$(\rho^\ell(S^a)f)(z) = \frac{s_a(z - \ell\eta)f(z + \eta) - s_a(-z - \ell\eta)f(z - \eta)}{\theta_{11}(2z; \tau)}, \quad (\text{A.5})$$

where

$$\begin{aligned}
s_0(z) &= \theta_{11}(\eta; \tau)\theta_{11}(2z; \tau), & s_1(z) &= \theta_{10}(\eta; \tau)\theta_{10}(2z; \tau), \\
s_2(z) &= i\theta_{00}(\eta; \tau)\theta_{00}(2z; \tau), & s_3(z) &= \theta_{01}(\eta; \tau)\theta_{01}(2z; \tau).
\end{aligned}$$

These representations reduce to the usual spin ℓ representations of $U(sl(2))$ for $J_{\alpha\beta} \rightarrow 0$ ($\eta \rightarrow 0$).

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